The Load Supported by Small Floating Objects

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We consider the equilibrium flotation of a two-dimensional cylinder and a sphere at an interface between two fluids. We give conditions on the density and radius of these objects for them to be able to float at the interface and discuss the role played by the contact angle in determining these conditions. For cylinders with a small radius, we find that the maximum density is independent of contact angle but that, for spheres, the contact angle enters at leading order in the particle radius. Our theoretical predictions are in agreement with experimental results.

Introduction

Small, dense objects are apparently able to violate Archimedes' principle: they can float at the surface of a liquid despite being many times more dense than the liquid. Of course, this feat is only possible because the surface tension of the liquid provides a vertical force that helps counteract the object's excess weight. In fact, the force provided by surface tension is precisely equal to the weight of liquid that is displaced in the meniscus around the edge of the object: Archimedes' principle modified to account for surface tension.[1] As well as being used by water-walking insects to avoid drowning,[2−4] this supporting force also has applications in self-assembly technologies where solid components are supported at interfaces and then spontaneously come into contact via capillary interactions.[5] In this article, we consider the question of how dense small objects can become and still float at an interface. We consider, in particular, the flotation of a two-dimensional cylinder and a sphere and give the maximum density that these objects can have and still float.

A question of particular interest is how the surface properties of an object affect its load-supporting properties. Previous work has assumed that superhydrophobic surfaces should be able to support significantly larger loads relative to those supported by surfaces with slightly smaller contact angles.[3] We find this to be the case for small floating spheres but *not* for two-dimensional cylinders.

Flotation of a Horizontal Cylinder

We begin by considering the equilibrium of a cylinder of density ρ_s, radius r_0, and contact angle θ at the interface between two fluids of density ρ_A and ρ_B, with $\rho_A < \rho_B$ (as depicted in Figure 1). The interface has a tension, γ_{AB}, associated with it. Balancing the weight per unit length of the cylinder with the restoring forces arising from surface tension and the Archimedes upthrust of fluid B on the object, we find that equilibrium requires

$$\pi(\rho_s - \rho_A)r_0^2 g = 2\gamma_{AB}\sin\phi + (\rho_B - \rho_A)g r_0^2\left(-2\frac{h_*}{r_0}\sin\psi + \psi - \sin\psi\cos\psi\right) \quad (1)$$

Here the angular position of the contact line, ψ, the inclination

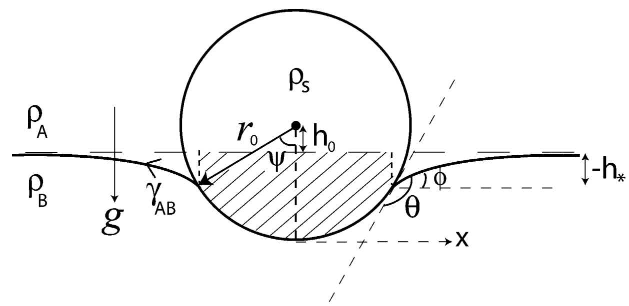

Figure 1. Dimensional notation used for the problem of a single cylinder floating at the interface between two fluids. Dimensionless variables are represented by uppercase letters in the text.

of the interface to the horizontal, ϕ, and the height of the contact line *above* the undeformed interface, h_*, determine the position of the cylinder at the interface. The first term on the right-hand side of eq 1 is the vertical component of the surface tension acting on the cylinder. The second term is the vertical force provided by the Archimedean upthrust on the cylinder, which is equal to the weight of fluid B displaced by the hatched region in Figure 1, as shown by Keller.[1]

We are interested in the interplay between surface tension and gravity, and so it is natural to use the *capillary length*, $\ell_c \equiv (\gamma_{AB}/(\rho_B - \rho_A)g)^{1/2}$ to nondimensionalize the lengths in eq 1. We use uppercase letters to denote nondimensional lengths, so that $R_0 \equiv r_0/\ell_c$, $H_* \equiv h_*/\ell_c$, and so on. Letting $B \equiv R_0^2$ and $D \equiv (\rho_s - \rho_A)/(\rho_B - \rho_A)$, eq 1 then becomes

$$\pi DB = 2\sin\phi + B\left(-2\frac{H_*}{R_0}\sin\psi + \psi - \sin\psi\cos\psi\right) \quad (2)$$

Equation 2 may be viewed as an equation giving the cylinder density, D, as a function of the position of the cylinder, which is parametrized by ϕ, ψ, and H_*. Using the geometrical relationship $\phi = \theta + \psi - \pi$, we may eliminate ψ from eq 2 in favor of ϕ. The quantities H_* and ϕ may also be related. In two dimensions, the Laplace−Young equation for the meniscus profile, $H(X)$, takes the simple form

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(1) Keller, J. B. *Phys. Fluids* **1998**, *10*, 3009−3010.
(2) Bush, J. W. M.; Hu, D. L. *Annu. Rev. Fluid Mech.* **2006**, *38*, 339−369.
(3) Gao, X.; Jiang, L. *Nature* **2004**, *432*, 36.
(4) Hu, D. L.; Chan, B.; Bush, J. W. M. *Nature* **2003**, *424*, 663−666.
(5) Whitesides, G. M.; Grzybowski, B. *Science* **2002**, *295*, 2418−2421.

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$$\frac{H_{XX}}{(1 + {H_X}^2)^{3/2}} = H \tag{3}$$

where $(\ )_X$ denotes differentiation with respect to X. This may be integrated once[6] to give the horizontal force balance condition

$$\cos \phi = 1 - {H_*}^2/2 \tag{4}$$

which may be used to eliminate H_* in favor of ϕ. With H_* and ψ eliminated from eq 2, we can then plot the function $D(\phi)$, as shown in Figure 2. We note, as Rappachietta et al.[7] did, that this shows that there is a maximum density, D_{max}, above which a cylinder cannot be in equilibrium and must therefore sink.

Previously,[7,8] it has been noted that, as the radius of an object decreases, the density it can have without sinking increases. In Figure 3, we quantify this statement by plotting D_{max} as a function of Bond number for different values of the contact angle θ. These results show good agreement between the theoretical calculations and experiments (see Materials and Methods section). We also note that the value of D_{max} does, for these values of B, depend on the value of θ.

By considering the behavior of eq 2 for $B \ll 1$, we find that

$$D_{max} \sim 2/(\pi B) \tag{5}$$

provided that $\theta > \pi/2$. Note that this is independent of the precise value of θ. This is simple to understand physically since, for $B \ll 1$, the force contribution from buoyancy is small compared to that of surface tension. The cylinder is therefore able to support the largest load when the meniscus is vertical ($\phi = \pi/2$) and the force per unit length that surface tension then provides is precisely γ, independently of θ. Figure 4 confirms the asymptotic result of eq 5 for $B \ll 1$ by plotting the computed values of D_{max} for two very different values of θ (both with $\theta > \pi/2$). For $\theta < \pi/2$, the meniscus cannot come close to vertical because this would require $\psi > \pi$. We therefore find that, for $B \ll 1$ with $\theta < \pi/2$,

$$D_{max} \sim 2 \sin \phi/(\pi B) \tag{6}$$

Materials and Methods

In the experiments to measure the maximum density of horizontal cylinders that can float at an air−water interface, we used hollow glass cylinders with lengths, L, much greater than their respective radii (typically, $L > 30R$) to minimize the three-dimensional effects. The cylinder radius ranged between 1.5 and 3.6 mm. The surfaces of the cylinders were coated with different materials, allowing us to investigate four different equilibrium contact angle conditions. We obtained $\theta = 72°$ by spraying a commercial nitrocellulose lacquer on the cylinder. By dipping the cylinder into molten candillela wax and paraffin wax, we obtained the surface contact angles $\theta = 93°$ and $\theta = 104°$, respectively. By spraying a mixture of chloroform and melted alkyl ketene dimer (AKD) on the cylinder surface,[9] we increased θ to 143°. This high contact angle is attributed to the microscopic roughness of the AKD surface, which has a contact angle of 109° on smooth surfaces.[10]

To find the maximum load that can be supported on the interface, we varied the cylinder weight by inserting different numbers of metal wires into the hollow cavity of the cylinder and sealing with a semi-flexible polymer. By iterating between the weights that float and sink, the maximum density of a cylinder that floats was obtained for each contact angle condition and for a given cylinder radius. Figure 5 shows an AKD-coated cylinder that is only just able to float on water, as evidenced by the highly curved meniscus.

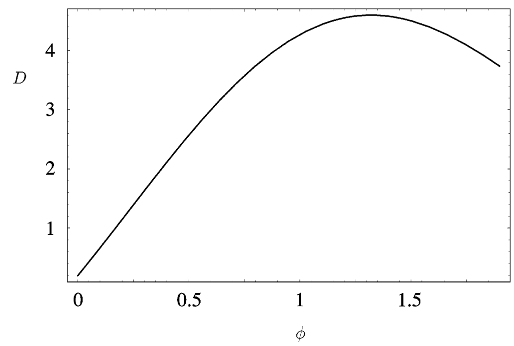

Figure 2. Variation of D as the interfacial inclination, ϕ, is varied. At some critical value of ϕ, the cylinder density D is maximized, and cylinders with a density larger than this must sink. Here, $B = 0.25$ and $\theta = 2\pi/3$.

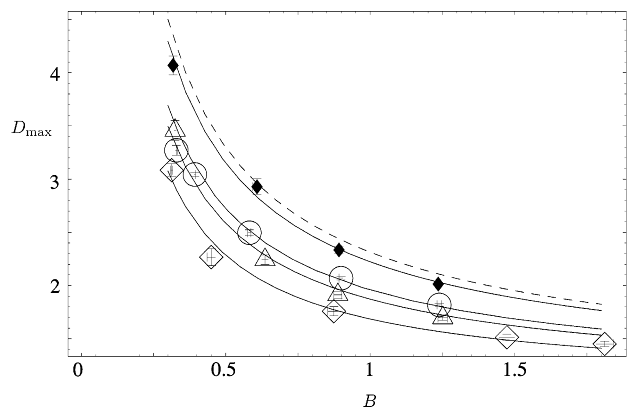

Figure 3. Dependence of D_{max} on the Bond number, $B = {R_0}^2$, for cylinders with different values of θ. Theoretical predictions (solid lines) compare favorably with the experimental results (points). (◇) $\theta = 72°$; (△) $\theta = 93°$; (○) $\theta = 104°$; (◆) $\theta = 143°$. The dashed line shows the theoretical prediction for $\theta = 180°$.

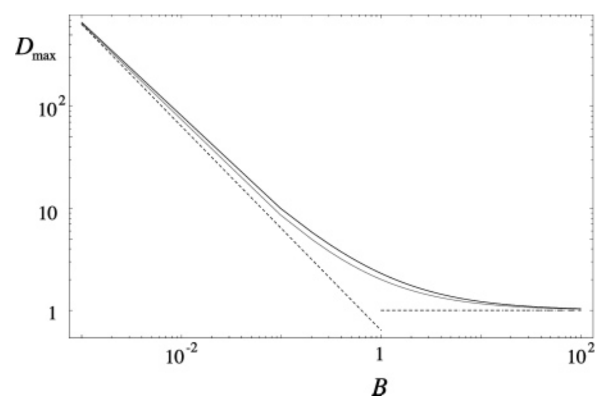

Figure 4. Dependence of D_{max} on the Bond number, $B \equiv {R_0}^2$, for cylinders with two different values of θ over a wide range of Bond numbers. Theoretical predictions for $\theta = 109°$ (gray) and $\theta = 167°$ (black) compare well with the asymptotic results (eq 5) for $B \ll 1$ and $D_{max} \sim 1$ for $B \gg 1$ (dashed lines).

Flotation of a Sphere

The preceding analysis and physical argument suggest that the contact angle has only a limited influence on the weight or density of an object that can float at an interface, provided that $\theta > \pi/2$ and that the object is sufficiently small. The complete

(6) Mansfield, E. H.; Sepangi, H. R.; Eastwood, E. A. *Philos. Trans. R. Soc. London, Ser. A* **1997**, *355*, 869−919.
(7) Rapacchietta, A. V.; Neumann, A. W.; Omenyi, S. N. *J. Colloid Interface Sci.* **1977**, *59*, 541−554.
(8) Rapacchietta, A. V.; Neumann, A. W. *J. Colloid Interface Sci.* **1977**, *59*, 555−567.
(9) Torkkeli, A.; Saarilahti, J.; Haara, A.; Harma, H.; Soukka T.; Tolonen, P. *Proceedings of the 14th IEEE International Conference on MEMS*, Interlaken, Switzerland, 2001.
(10) Onda, T.; Shibuichi, S.; Satoh, N.; Tsujii, K. *Langmuir* **1996**, *12*, 2125−2127.

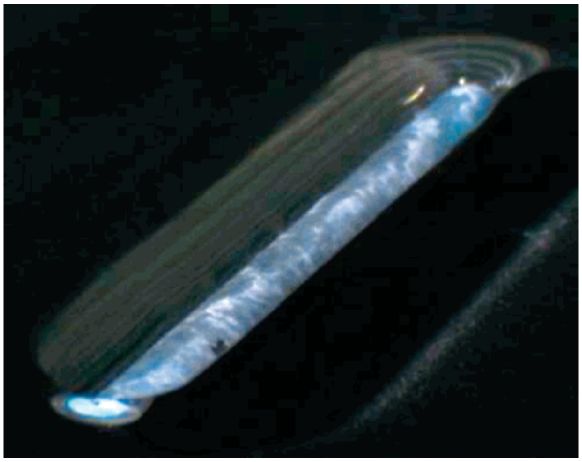

Figure 5. A floating cylinder coated with AKD. The cylinder's radius is 3.1 mm and its mass is 4.7 g.

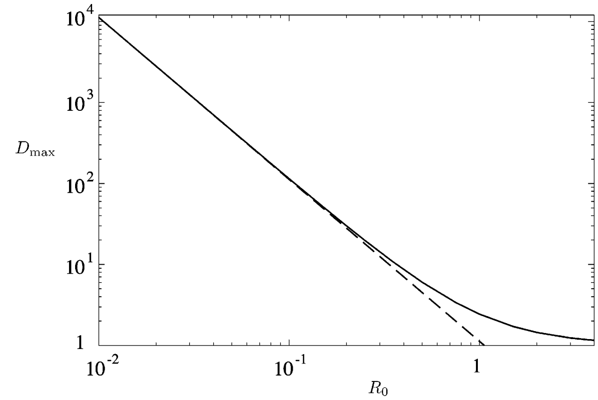

Figure 6. The numerically determined dependence of the maximum sphere density, D_{max}, on the radius for a sphere with contact angle $\theta = 2\pi/3$. The dashed line shows the asymptotic result (eq 10) for this value of θ.

picture, however, is more complicated. In this section, we show that, even when $\theta > \pi/2$, the precise value of the contact angle can be of some consequence by considering the conditions required for a sphere to float in equilibrium.

Balancing the sphere's weight with the vertical contribution of the surface tension and the Archimedean upthrust, we find that the vertical force balance condition has the nondimensional form:

$$\frac{4}{3}DR_0^{\ 3} = 2R_0 \sin\psi \sin\phi + R_0^{\ 3}\left(-\frac{H_*}{R_0}\sin^2\psi + \frac{2}{3} - \cos\psi + \frac{1}{3}\cos^3\psi\right) \quad (7)$$

We may once again consider eq 7 as an equation for the density of the sphere, D, as a function of its position, parametrized by H_*, ϕ, and ψ. We may again use the geometrical relationship $\phi = \theta + \psi - \pi$ to eliminate ψ from eq 7. There is no analogue, however, of eq 4, and so, to relate ϕ and H_*, we must numerically determine the meniscus shape $H(R)$, where R is the radial coordinate. This was done by solving the nondimensionalized Laplace−Young equation in an axisymmetric geometry:[11]

$$H = \frac{1}{R}\left(\frac{RH_R}{(1 + H_R^{\ 2})^{1/2}}\right)_R \quad (8)$$

where $(\ )_R$ denotes differentiation with respect to R. Equation 8 is to be solved with the boundary conditions

$$H_R(R_0 \sin\psi) = \tan\phi, \ H(\infty) = 0 \quad (9)$$

Equations 8 and 9 were solved numerically using the MATLAB routine bvp4c to determine H_* for a given ϕ and thereby determine $D(\phi)$, as has been done previously.[8] In Figure 6 we present, for the first time, a plot of D_{max} over a wide range of values of R_0.

By maximizing the density as ϕ varies in eq 7 for $R_0 \ll 1$, we find that

$$D_{max} \sim \frac{3}{4R_0^{\ 2}}(1 - \cos\theta) \quad (10)$$

This corresponds very well with the numerical results shown in Figure 6 and works equally well for other values of θ.

The result in eq 10 is interesting because it is the leading order behavior of D_{max} for $R_0 \ll 1$ and yet contains a θ dependence, even though $\theta > \pi/2$. This is in stark contrast to what we found for long cylinders earlier, where a dependence on θ is only seen in the higher order corrections. Additionally, this maximum is attained when the meniscus makes an angle $\phi = \theta/2$ with the horizontal, in contrast to the near vertical deformation ($\phi = \pi/2$) that is typical of cylinder sinking. Both of these differences are consequences of the geometry in this situation: as the position of the contact line is varied, there is a competition between maximizing the inclination of the meniscus at the contact line (large values of ψ) and maximizing the contact line length (requiring $\psi \approx \pi/2$). This competition leads to the selection of an intermediate value of ψ at which D is maximized, and so introduces some dependence on θ. It is therefore entirely natural for the corresponding value of D_{max} to depend on θ.

Discussion

We conclude that having a very large contact angle does not enable a cylinder to support significantly larger loads than can be obtained with cylinders with $\theta \geq \pi/2$. This appears to contradict the results of experiments comparing the load-bearing capacity of water strider legs (for which $\theta = 167°$) with glass fibers coated to give $\theta = 109°$.[3] In these experiments, it was found that the water strider leg could support a force of 152 dynes, while the glass fiber could support a load of just 19 dynes. Because these fibers were of similar dimensions (length 9 mm and diameter ~90 μm), it was concluded that the difference in load bearing capacity is purely due to their different wetting properties. Modeling the leg as a long, inflexible cylinder, we find that both leg and fiber should support forces of around ~140 dynes, which is in reasonable agreement with the experiment on a water strider leg but not with the experiment on a glass fiber. This suggests that there may have been an imperfection in Gao and Jiang's modification of the surface of the glass fiber. They used a self-assembling monolayer of heptadecafluorodecyltrimethoxysilane, which is known to make a contact angle of 109° with water on a "flat" surface. Our theory suggests that their observation of a maximum supportable load of 19 dynes is consistent with a fiber of the same dimensions, but with a contact angle of 9°, which is a typical value for water on untreated glass.

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(11) Finn, R. *Equilibrium Capillary Surfaces*; Springer: Berlin, 1986.